MATH2050B 1920 HW7

TA's solutions[1] to selected problems

Q1.

(a) In terms of sequences, state the density result for $\mathbb{Q}$. Do the same for $\mathbb{R} \setminus \mathbb{Q}$.

(b) State the sequential criterion for $\lim_{x\to x_0} = l$, $+\infty$ or $-\infty$.

(c) State Cauchy criterion for sequence.

These results may be helpful for **Q2, Q3, Q4** below.

Solution.

(a) (Density of $\mathbb{Q}$) For any real number x, there is a sequence of rational numbers $(q_n)_{n=1}^{\infty}$ s.t. $q_n \neq x$ for all n and $q_n \to x$.

(Density of $\mathbb{R}\setminus\mathbb{Q}$) For any real number x, there is a sequence of irrational numbers $(r_n)_{n=1}^{\infty}$ s.t. $r_n \neq x$ for all n and $r_n \to x$.

(b) $\lim_{x\to x_0} f(x) = l$ if and only if $f(x_n) \to l$ for every sequence $(x_n)_{n=1}^{\infty}$ with $x_n \neq x_0$ for all n and $x_n \to x_0$.

The cases for $+\infty$ and $-\infty$ are similar.

(c) A sequence of real numbers $(x_n)_{n=1}^{\infty}$ is convergent if and only if it is Cauchy.

Q2. Let $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x) = \begin{cases} 6x - 8 & \forall x \in \mathbb{Q} \\ \frac{7}{x} - 7 & \forall x \notin \mathbb{Q} \end{cases}$$

Find all x_0 for which $\lim_{x\to x_0} g(x)$ exists in $\mathbb{R}$.

Solution. Claim. $\lim_{x\to x_0} g(x)$ exists in $\mathbb{R}$ iff $6x_0 - 8 = \frac{7}{x_0} - 7$.

If $\lim_{x\to x_0} g(x) = L \in \mathbb{R}$. Note that x_0 cannot be 0, because $\lim_{x\to 0} g(x)$ does not exist. Choose a sequence of rationals (q_n) and irrationals (r_n) s.t. $q_n, r_n \to x_0$. By **Q1(c)**, $g(q_n), g(r_n) \to L$. We have

$$\lim_{n\to\infty} g(q_n) = \lim_{n\to\infty} 6q_n - 8 = L = \lim_{n\to\infty} \frac{7}{r_n} - 7 = \lim_{n\to\infty} g(r_n).$$

This gives $6x_0 - 8 = \frac{7}{x_0} - 7$.

For the converse, suppose $6x_0 - 8 = \frac{7}{x_0} - 7 = L \in \mathbb{R}$. Then $x_0 \neq 0$. Let $\epsilon > 0$.

Note that the functions $6x - 8$, $\frac{7}{x} - 7$ are continuous at $x_0 \neq 0$, so there is $\delta > 0$ s.t. for all x with $0 < |x - x_0| < \delta$, we have

$$|6x - 8 - L| < \epsilon, \quad |\frac{7}{x} - 7 - L| < \epsilon.$$

It follows that for all x with $0 < |x - x_0| < \delta$, $|g(x) - L| < \epsilon$. The claim is proved.

[1] please kindly send an email to `nclliu@math.cuhk.edu.hk` if you have spotted any typo/error/mistake.

To find all the x_0 s.t. $\lim_{x\to x_0} g(x)$ exists in $\mathbb{R}$, it only needs to find x s.t. $6x - 8 = \frac{7}{x} - 7$. $x = -1$ or $\frac{7}{6}$.

Q3. Let $h : \mathbb{R} \to \mathbb{R}$ and $x_0 \in \mathbb{R}$, $l \in \mathbb{R}$. Show that $\lim_{x\to x_0} f(x) = l$ iff $\forall \epsilon > 0 \exists \delta > 0$ s.t. $|f(x) - f(n)| < \epsilon$ whenever $x, n, \in V_\delta(x_0) \setminus \{x_0\}$.

Solution. Suppose $\lim_{x\to x_0} f(x) = l \in \mathbb{R}$. Let $\epsilon > 0$, there is δ s.t. for all $x \in V_\delta(x_0) \setminus \{x_0\}$,

$$|f(x) - l| < \frac{\epsilon}{2}.$$

So for all $x, n \in V_\delta(x_0) \setminus \{x_0\}$,

$$|f(x) - f(n)| \le |f(x) - l| + |f(n) - l| < \epsilon.$$

For the converse, given any $(x_n)_{n=1}^\infty$, $x_n \neq x_0$ and $x_n \to x_0$. We want to show that $(f(x_n))_{n=1}^\infty$ is Cauchy. Let $\epsilon > 0$, then there is $\delta > 0$ s.t. $|f(x) - f(n)| < \epsilon$ whenever $x, n \in V_\delta(x_0) \setminus \{x_0\}$.

For the positive number $\delta > 0$, there is $N \in \mathbb{N}$ s.t. $|x_n - x_0| < \delta$ for all $n > N$. Since $x_n \neq 0$, it follows that x_n, $x_m \in V_\delta(x_0) \setminus \{x_0\}$ for all $n, m > N$. Thus $|f(x_n) - f(x_m)| < \epsilon$ for all $n, m > N$.

By assumption, $\lim_{n\to\infty} f(x_n)$ is independent of the choice of (x_n), that is, if (y_n) is another sequence with $y_n \neq x_0$, $y_n \to x_0$, then $\lim_n f(x_n) = \lim_n f(y_n)$. (**Please check**)

Hence $\lim_{x\to x_0} f(x)$ exists.

Q4. Let $f : \mathbb{R} \to \mathbb{R}$ be $\mathbb{Q}$-linear in the sense that $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)\, \forall \alpha, \beta \in \mathbb{Q}$ and $\forall x, y \in \mathbb{R}$. Suppose $\lim_{x\to 0} f(x) = L \in \mathbb{R}$. Show that f is continuous at any $x_0 \in \mathbb{R}$ in the sense that $\lim_{x\to x_0} f(x) = f(x_0)$.

Solution. We check $\lim_{x\to x_0} f(x) = f(x_0)$.

Special case. $x_0 = 0$.

By linearity, $f(0) = 0$. Also, it can be readily checked that $L = 0$. (choose a sequence of rationals $(q_n)_{n=1}^\infty$, $q_n \neq 0$ and $q_n \to 0$. By assumption $f(q_n) \to L$. On the other hand, by linearity, $f(q_n) = q_n f(1) \to 0$)

General case. $x_0 \in \mathbb{R}$. By the special case we have:

$$\lim_{x\to x_0} f(x) - f(x_0) = \lim_{x\to x_0} f(x - x_0) = \lim_{y\to 0} f(y) = 0.$$

Q5. Let $I = (a, b) \subset (0, \infty)$ be an interval of length 1 (or finite length). Let $n \in \mathbb{N}$ and

- $Z_n = \{m \in \mathbb{N} : \frac{m}{n} \in I\}$
- $Y_n = \{x \in I \cap \mathbb{Q} : x = \frac{m}{n}$ with some $m \in Z_n\}$
- $B_n = \{x \in I \cap \mathbb{Q} : x = \frac{m}{n}$, $\gcd(m, n) = 1$, with some $m \in Z_n\}$

Show that

1) Z_n is bounded and is a finite set;

2) Y_n and B_n are finite sets;

3) $I \cap \mathbb{Q} = \cup_{n\in\mathbb{N}} B_n$;

4) Let $x_0 \in \mathbb{R}^+ \setminus \mathbb{Q}$ (positive irrational), $I := (x_0 - \frac{1}{2}, x_0 + \frac{1}{2}) \cap \mathbb{R}^+$ and let $N \in \mathbb{N}$,

$$\delta = \min\{\frac{1}{2}, \text{dist}(x_0, \cup_{n=1}^N B_n)\}$$

where B_n is defined as before. Then $\delta > 0$ (Why?) and if $0 < x \in V_\delta(x_0) \setminus \mathbb{Q}$ and $x = \frac{m}{n}$ in canonical representation, then $n > N$.

(In the above $0 < x \in V_\delta(x_0) \setminus \mathbb{Q}$ should be $0 < x \in V_\delta(x_0) \cap \mathbb{Q}$)

Solution. (1) : I is a bounded interval. Fix $n \in \mathbb{N}$, then the sequence $(\frac{m}{n})_{m=1}^\infty$ is increasing, unbounded. So we find a large M s.t. $b < \frac{M}{n}$. This shows $Z_n \subset \{1, 2, \dots, M-1\}$.

(2) : The map $Y_n \to Z_n$, $x = \frac{m}{n} \mapsto m$ is injective: if $m_1 = m_2$, then $\frac{m_1}{n} = \frac{m_2}{n}$. So Y_n is finite.

(3) : $\cup_{n\in\mathbb{N}} B_n \subset I \cap \mathbb{Q}$ is by definition. For the converse, if $x \in I \cap \mathbb{Q}$, then $x \in \mathbb{Q}$. Write $x = \frac{p}{q}$ in canonical representation. Then $p \in Z_q$, $x \in B_q$. Thus $I \cap \mathbb{Q} \subset \cup_{n\in\mathbb{N}} B_n$.

(4) : To see $\delta > 0$, note $\cup_{n=1}^N B_n$ is a finite set not containing x_0. Thus $\delta > 0$. If $0 < x = \frac{m}{n} \in V_\delta(x_0) \cap \mathbb{Q}$, then $x \notin \cup_{j=1}^N B_j$, so $x \in \cup_{j=N+1}^\infty B_j$. Hence $n > N$.

Q6. The Thomae function $f : (0, \infty) \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \frac{1}{n} & x = \frac{m}{n} \text{ (canonical representation) }, x \in \mathbb{Q} \cap \mathbb{R}^+ \\ 0 & x \in \mathbb{R}^+ \setminus \mathbb{Q} \end{cases}$$

is continuous at any $x_0 \in \mathbb{R}^+ \setminus \mathbb{Q}$.

Solution. Let $x_0 \in \mathbb{R}^+ \setminus \mathbb{Q}$. Let $\epsilon > 0$, find N s.t. $\frac{1}{N} < \epsilon$, and set $\delta = \min\{\frac{1}{2}, \text{dist}(x_0, \cup_{n=1}^N B_n)\}$ as in **Q5**. Then $\delta > 0$.

For all $x \in V_\delta(x_0)$, we have either $x \in \mathbb{Q}$ or $x \notin \mathbb{Q}$.

- $x \in \mathbb{Q} \Rightarrow x = \frac{m}{n}$ in canonical representation. By **Q5 (4)**, $n > N$, so $|f(x)| = \frac{1}{N} < \epsilon$.
- $x \notin \mathbb{Q} \Rightarrow |f(x)| = 0 < \epsilon$.

Hence f is continuous at x_0.

Remark. The Thomae function is discontinuous at every $q \in \mathbb{Q} \cap \mathbb{R}^+$. To see this note $f(q) > 0$, take a sequence of irrationals $(r_n)_{n=1}^\infty$, $r_n \to q$, and appeal to sequential criteria.